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INVARIANT DIFFERENTIAL OPERATORS ON
NON-REDUCTIVE HOMOGENEOUS SPACES

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by

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ABSTRACT

A systematic exposition is given of the theory of invariant differential operators on a not necessarily reductive homogeneous space. This exposition is modelled on Helgason's treatment of the general reductive case and the special non-reductive case of the space of horocycles. As a final application the differential operators on (not a priori reductive) isotropic pseudo-Riemannian spaces are characterized.

KEY WORDS & PHRASES: *invariant differential operators; non-reductive homogeneous spaces; space of horocycles; isotropic pseudo-Riemannian spaces*

1. INTRODUCTION

Let G be a Lie group and H a closed subgroup. Let \mathfrak{g} and \mathfrak{h} denote the corresponding Lie algebras. Suppose that the coset space G/H is *reductive*, i.e., there is a complementary subspace \mathfrak{m} to \mathfrak{h} in \mathfrak{g} such that $\text{Ad}_G(H)\mathfrak{m} \subset \mathfrak{m}$. Let $\mathbb{D}(G/H)$ denote the algebra of G -invariant differential operators on G/H . The main facts about $\mathbb{D}(G/H)$ are summarized below (cf. HELGASON [3, Ch. III], [4, Cor. X.2.6, Theor. X.2.7], [6, §2]).

Let $\mathbb{D}(G)$ be the algebra of left invariant differential operators on G , $\mathbb{D}_H(G)$ the subalgebra of operators which are right invariant under H and $S(\mathfrak{g})$ the complexified symmetric algebra over \mathfrak{g} . Let $\lambda: S(\mathfrak{g}) \rightarrow \mathbb{D}(G)$ denote the symmetrization mapping. $I(\mathfrak{m})$ denotes the set of $\text{Ad}_G(H)$ -invariants in $S(\mathfrak{m})$. Then

$$(1.1) \quad \mathbb{D}_H(G) = \mathbb{D}(G)\mathfrak{h} \cap \mathbb{D}_H(G) \oplus \lambda(I(\mathfrak{m})).$$

Let $\pi: G \rightarrow G/H$ be the natural mapping. Let $C_H^\infty(G)$ consist of the C^∞ -functions on G which are right invariant under H . Write $\tilde{f} := f \circ \pi$ ($f \in C^\infty(G/H)$) and $(D_u f)^\sim := u\tilde{f}$ ($f \in C^\infty(G/H)$, $u \in \mathbb{D}_H(G)$). Then $D_u \in \mathbb{D}(G/H)$.

THEOREM 1.1. *The mapping $u \rightarrow D_u$ is an algebra homomorphism from $\mathbb{D}_H(G)$ onto $\mathbb{D}(G/H)$ with kernel $\mathbb{D}(G)\mathfrak{h} \cap \mathbb{D}_H(G)$. The mapping $P \rightarrow D_{\lambda(P)}: I(\mathfrak{m}) \rightarrow \mathbb{D}(G/H)$ is a linear bijection.*

Theorem 1.1. is of basic importance for the analysis on symmetric spaces. In particular, it can be shown that $\mathbb{D}(G/H)$ is commutative if G/H is a pseudo-Riemannian symmetric space which admits a relatively invariant measure. In its most general form this result was proved by DUFLLO [1] in an algebraic way. G. van Dijk kindly communicated a short analytic proof of Duflo's result to me (unpublished). In [1] DUFLLO used generalizations of (1.1) and Theorem 1.1 to the case of homogeneous line bundles over G/H . These can be proved by only minor changes of Helgason's original proofs.

There exist non-reductive coset spaces G/H for which $\mathbb{D}(G/H)$ is still commutative. For instance, let G be a connected real semisimple Lie group and let M and N be the usual subgroups of G . Then G/MN is the space of horocycles and $\mathbb{D}(G/MN)$ is commutative. In order to prove this, formula (1.1)

and Theorem 1.1 have to be adapted to the non-reductive case. While HELGASON [5, §4], [6, §3] has done this in an ad hoc way for the special coset spaces under consideration, it is the purpose of the present note to give a more systematic exposition of the theory of $\mathbb{D}(G/H)$ for a not necessarily reductive coset space.

Furthermore, following Duflo, the theory will be developed for invariant differential operators on homogeneous line bundles over G/H . As a final application we will characterize $\mathbb{D}(G/H)$ for isotropic pseudo-Riemannian symmetric spaces G/H without a priori knowledge that G/H is reductive. Throughout HELGASON [4] will be our standard reference.

2. DEVELOPMENT OF THE GENERAL THEORY

Let G be a Lie group with Lie algebra \mathfrak{g} . For $X \in \mathfrak{g}$ define the vector field \tilde{X} on G by

$$(2.1) \quad (\tilde{X}f)(g) := \left. \frac{d}{dt} f(g \exp tX) \right|_{t=0}, \quad f \in C^\infty(G), g \in G.$$

Then the mapping $X \rightarrow \tilde{X}$ is an isomorphism from \mathfrak{g} onto the Lie algebra of left invariant vector fields on G . Throughout this section let X_1, \dots, X_n be a fixed basis of \mathfrak{g} .

For a finite-dimensional real vector space V the symmetric algebra $S(V)$ is defined as the algebra of all polynomials with complex coefficients on V^* , the dual of V . Let $S^m(V)$ respectively $S_m(V)$ ($m = 0, 1, 2, \dots$) denote the space of homogeneous polynomials of degree m on V^* , respectively of polynomials of degree $\leq m$ on V^* . Thus $S^m(G)$ is spanned by the monomials $X_{i_1} X_{i_2} \dots X_{i_m}$ ($i_1, \dots, i_m \in \{1, \dots, n\}$).

Let $\mathbb{D}(G)$ be the algebra of left invariant differential operators on G with complex coefficients. For $P \in S(\mathfrak{g})$ define an operator $\lambda(P)$ on $C^\infty(G)$ by

$$(2.2) \quad (\lambda(P)f)(g) := P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) f(g \exp(t_1 X_1 + \dots + t_n X_n)) \Big|_{t_1 = \dots = t_n = 0},$$

where

$$P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) := \frac{\partial^m}{\partial t_{i_1} \dots \partial t_{i_m}} \quad \text{for } P = X_{i_1} \dots X_{i_m}.$$

It is proved in [4, Prop. II.1.9 and p. 392] that:

PROPOSITION 2.1. *The mapping $P \rightarrow \lambda(P)$ is a linear bijection from $S(g)$ onto $\mathbb{D}(G)$. It satisfies:*

$$(2.3) \quad \lambda(Y^m) = \tilde{Y}^m, \quad Y \in g;$$

$$(2.4) \quad \lambda(Y_1 \dots Y_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \tilde{Y}_{\sigma(1)} \dots \tilde{Y}_{\sigma(m)}, \quad Y_1, \dots, Y_m \in g.$$

The definition of λ is independent of the choice of the basis of g .

The mapping λ is called *symmetrization*. The Lie algebra g is embedded as a subspace of $\mathbb{D}(G)$ under the mapping $X \rightarrow \tilde{X}$. Any homomorphism from g to g uniquely extends to a homomorphism from $\mathbb{D}(G)$ to $\mathbb{D}(G)$ and any linear mapping from g to g uniquely extends to a homomorphism from $S(g)$ to $S(g)$. In particular, for $g \in G$, the automorphism $\text{Ad}(g)$ of g uniquely extends to automorphisms of both $S(g)$ and $\mathbb{D}(G)$ and

$$(2.5) \quad \lambda(\text{Ad}(g)P) = \text{Ad}(g)\lambda(P), \quad P \in S(g), g \in G.$$

For $g, g_1 \in G$, $f \in C^\infty(G)$, $D \in \mathbb{D}(G)$ write

$$f^{R(g)}(g_1) := f(g_1 g); \quad D^{R(g)} f := (Df^{R(g^{-1})})^{R(g)}.$$

Then

$$(2.6) \quad \text{Ad}(g)D = D^{R(g^{-1})}, \quad D \in \mathbb{D}(G), g \in G.$$

Let H be a closed subgroup of G and let \mathfrak{h} be the corresponding subalgebra. Let \mathfrak{m} be a subspace of g complementary to \mathfrak{h} . Let X_1, \dots, X_r be a basis of \mathfrak{m} and X_{r+1}, \dots, X_n a basis of \mathfrak{h} . Let χ be a character of H , i.e. a continuous homomorphism from H to the multiplicative group $\mathbb{C} \setminus \{0\}$. Throughout this section, H , \mathfrak{m} , the basis and χ will be assumed fixed.

Let $\pi: G \rightarrow G/H$ be the canonical mapping. Write $0 := \pi(e)$. Let

$$(2.7) \quad C_{H,\chi}^\infty(G) := \{f \in C^\infty(G) \mid f(gh) = f(g)\chi(h^{-1}), g \in G, h \in H\}.$$

Sometimes we will assume that χ has an extension to a character on G . This assumption clearly holds if $\chi \equiv 1$ on H , but it does not hold for general H and χ . For instance, if $G = \mathrm{SU}(2)$ or $\mathrm{SL}(2, \mathbb{R})$ and $H = \mathrm{SO}(2)$ then nontrivial characters on H do not extend to characters on G .

If χ extends to a character on G then we define a linear bijection $f \rightarrow \tilde{f}: C^\infty(G/H) \rightarrow C_{H,\chi}^\infty(G)$ by

$$(2.8) \quad \tilde{f}(g) := f(\pi(g))\chi(g^{-1}), \quad g \in G.$$

LEMMA 2.2. Let $P \in S(m)$. If $\lambda(P)f = 0$ for all $f \in C_{H,\chi}^\infty(G)$ then $P = 0$.

PROOF. For each $f \in C^\infty(G/H)$ we can find $F \in C_{H,\chi}^\infty(G)$ such that

$$F(\exp(t_1 X_1 + \dots + t_r X_r)) = f(\exp(t_1 X_1 + \dots + t_r X_r).0)$$

for (t_1, \dots, t_r) in some neighbourhood of $(0, \dots, 0)$. Hence

$$\begin{aligned} 0 &= (\lambda(P)F)(e) = \\ &= P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}\right) f(\exp(t_1 X_1 + \dots + t_r X_r).0) \Big|_{t_1 = \dots = t_r = 0} \end{aligned}$$

for all $f \in C^\infty(G/H)$, so $P = 0$. \square

Let the differential of χ also be denoted by χ . Let $h^{\mathbb{C}}$ be the complexification of h . Let

$$(2.9) \quad h^\chi := \{X + \chi(X) \mid X \in h^{\mathbb{C}}\} \subset \mathbb{D}(G).$$

Clearly, $Df = 0$ if $f \in C_{H,\chi}^\infty(G)$ and $D \in h^\chi$. Let $\mathbb{D}(G)h^\chi$ be the linear span of all vw with $v \in \mathbb{D}(G)$, $w \in h^\chi$. Observe that, by Proposition 2.1, $\tilde{Y}_1 \dots \tilde{Y}_m \in \lambda(S_m(g))$ for $Y_1, \dots, Y_m \in g$. The following proposition was proved in [4, Lemma X.2.5] for $\chi \equiv 1$.

PROPOSITION 2.3. *There are the direct sum decompositions*

$$(2.10) \quad \lambda(S_m(g)) = \lambda(S_{m-1}(g))h^X \oplus \lambda(S_m(m))$$

and

$$(2.11) \quad \mathbb{D}(G) = \mathbb{D}(G)h^X \oplus \lambda(S(m)).$$

PROOF. First we prove by complete induction with respect to m that

$$\lambda(S_m(g)) \subset \lambda(S_{m-1}(g))h^X + \lambda(S_m(m)).$$

This clearly holds for $m = 0$. Suppose it is true for $m < d$. Let

$$P = X_1^{d_1} \dots X_n^{d_n}, \quad d_1 + \dots + d_n = d.$$

If $d_{r+1} + \dots + d_n = 0$ then $P \in S_d(m)$, so $\lambda(P) \in \lambda(S_d(m))$. If $d_{r+1} + \dots + d_n > 0$ then, by (2.4), $\lambda(P)$ is a linear combination of certain elements $\tilde{Y}_1 \dots \tilde{Y}_d$ with $Y_i \in \mathfrak{h}$ for at least one i , so

$$\lambda(P) \in \lambda(S_{d-1}(g))h^{\mathbb{C}} + \lambda(S_{d-1}(g)) \subset \lambda(S_{d-1}(g))h^X + \lambda(S_{d-1}(g)).$$

Now apply the induction hypothesis. This yields (2.10) and (2.11) (use Proposition 2.1) except for the directness.

To prove the directness of the sum (2.11), suppose that $P \in S(m)$ and $\lambda(P) \in \mathbb{D}(G)h^X$. Then $\lambda(P)f = 0$ for all $f \in C_{H,X}^{\infty}(G)$, so $P = 0$ by Lemma 2.2. \square

LEMMA 2.4. Let $D \in \mathbb{D}(G)$. Then $Df = 0$ for all $f \in C_{H,X}^{\infty}(G)$ if and only if $D \in \mathbb{D}(G)h^X$.

PROOF. Apply Proposition 2.3 and Lemma 2.2. \square

Let us define

$$(2.12) \quad \mathbb{D}_{H,X,\text{mod}}(G) := \{D \in \mathbb{D}(G) \mid \text{Ad}(h)D - D \in \mathbb{D}(G)h^X \text{ for all } h \in H\}.$$

This definition is motivated by the following lemma.

LEMMA 2.5. *Let $D \in \mathbb{D}(G)$. Then the following two statements are equivalent.*

- (i) $D \in \mathbb{D}_{H,\chi,\text{mod}}(G)$.
- (ii) $f \in C_{H,\chi}^\infty(G) \Rightarrow Df \in C_{H,\chi}^\infty(G)$.

PROOF. Let $D \in \mathbb{D}(G)$. If $f \in C_{H,\chi}^\infty(G)$, $h \in H$ then

$$(*) \quad (Df)^{R(h)} = D^{R(h)} f^{R(h)} = \chi(h^{-1}) D^{R(h)} f.$$

First assume (i). If $f \in C_{H,\chi}^\infty(G)$, $h \in H$ then $(D^{R(h)} - D)f = (\text{Ad}(h)D - D)f = 0$, so combination with (*) yields $(Df)^{R(h)} = \chi(h^{-1})Df$, i.e., $Df \in C_{H,\chi}^\infty(G)$. Conversely assume (ii). If $f \in C_{H,\chi}^\infty(G)$, $h \in H$ then $(Df)^{R(h)} = \chi(h^{-1})Df$, so combination with (*) yields $(D^{R(h)} - D)f = 0$. Hence $\text{Ad}(h)D - D = D^{R(h)} - D \in \mathbb{D}(G)h^\chi$ by Lemma 2.4. \square

From the preceding results the following theorem is now obvious.

THEOREM 2.6.

- (a) $\mathbb{D}_{H,\chi,\text{mod}}(G)$ is a subalgebra of $\mathbb{D}(G)$.
- (b) $\mathbb{D}(G)h^\chi$ is a two-sided ideal in $\mathbb{D}_{H,\chi,\text{mod}}(G)$.
- (c) There is the direct sum decomposition

$$(2.13) \quad \mathbb{D}_{H,\chi,\text{mod}}(G) = \mathbb{D}(G)h^\chi \oplus \lambda(S(m)) \cap \mathbb{D}_{H,\chi,\text{mod}}(G).$$

- (d) Define the mappings A and B by

$$\begin{aligned} u &\xrightarrow{A} u(\text{mod } \mathbb{D}(G)h^\chi) \xrightarrow{B} u \Big|_{C_{H,\chi}^\infty(G)} : \\ \lambda(S(m)) \cap \mathbb{D}_{H,\chi,\text{mod}}(G) &\xrightarrow{A} \mathbb{D}_{H,\chi,\text{mod}}(G) / \mathbb{D}(G)h^\chi \xrightarrow{B} \\ &\xrightarrow{B} \mathbb{D}_{H,\chi,\text{mod}} \Big|_{C_{H,\chi}^\infty(G)}. \end{aligned}$$

Then A is a linear bijection and B is an algebra isomorphism onto.

Define the mapping $\sigma: g \rightarrow m$ by

$$(2.14) \quad \sigma(X+Y) := X, \quad X \in \mathfrak{m}, Y \in \mathfrak{h}.$$

Consider $S(\mathfrak{m})$ as a subalgebra of $S(\mathfrak{g})$. Thus, if $P \in S(\mathfrak{m})$ and $h \in H$ then $\text{Ad}(h)P \in S(\mathfrak{g})$ and $\sigma \circ \text{Ad}(h)P \in S(\mathfrak{m})$ are well-defined. By an application of (2.4) we see that, if $Q \in S_{\mathfrak{m}}(\mathfrak{g})$ then

$$(2.15) \quad \lambda(\sigma Q - Q) \in \lambda(S_{\mathfrak{m}-1}(\mathfrak{g})) + \mathbb{D}(G)h^\chi.$$

Define the algebra

$$(2.16) \quad I_{\text{mod}}(\mathfrak{m}) := \{P \in S(\mathfrak{m}) \mid \sigma \circ \text{Ad}(h)P = P \text{ for all } h \in H\}.$$

LEMMA 2.7. *Let $P \in S(\mathfrak{m})$ such that $\lambda(P) \in \mathbb{D}_{H,\chi,\text{mod}}(G)$. Write $P = P^{\mathfrak{m}} + P_{\mathfrak{m}-1}$, where $P^{\mathfrak{m}} \in S^{\mathfrak{m}}(\mathfrak{m})$, $P_{\mathfrak{m}-1} \in S_{\mathfrak{m}-1}(\mathfrak{m})$. Then $P^{\mathfrak{m}} \in I_{\text{mod}}(\mathfrak{m})$.*

PROOF. $\lambda(\text{Ad}(h)P - P) \in \mathbb{D}(G)h^\chi$ by (2.12). Hence

$$\lambda(\text{Ad}(h)P^{\mathfrak{m}} - P^{\mathfrak{m}}) \in \lambda(S_{\mathfrak{m}-1}(\mathfrak{g})) + \mathbb{D}(G)h^\chi.$$

So

$$\lambda(\sigma \circ \text{Ad}(h)P^{\mathfrak{m}} - P^{\mathfrak{m}}) \in \lambda(S_{\mathfrak{m}-1}(\mathfrak{g})) + \mathbb{D}(G)h^\chi \subset \lambda(S_{\mathfrak{m}-1}(\mathfrak{m})) + \mathbb{D}(G)h^\chi,$$

where we used (2.16) and (2.10). By directness of the decomposition (2.10):

$$\sigma \circ \text{Ad}(h)P^{\mathfrak{m}} - P^{\mathfrak{m}} \in S_{\mathfrak{m}-1}(\mathfrak{m}).$$

Hence $\sigma \circ \text{Ad}(h)P^{\mathfrak{m}} - P^{\mathfrak{m}}$, being homogeneous of degree \mathfrak{m} , is the zero polynomial. \square

PROPOSITION 2.8. *If $\lambda(I_{\text{mod}}(\mathfrak{m})) \subset \mathbb{D}_{H,\chi,\text{mod}}(G)$ then*

$$\lambda(I_{\text{mod}}(\mathfrak{m})) = \lambda(S(\mathfrak{m})) \cap \mathbb{D}_{H,\chi,\text{mod}}(G)$$

and the mapping

$$D \mapsto D \Big|_{C_{H,\chi}^\infty(G)} : \lambda(I_{\text{mod}}(m)) \rightarrow \mathbb{D}_{H,\chi,\text{mod}}(G) \Big|_{C_{H,\chi}^\infty(G)}$$

is a linear bijection.

PROOF. Use complete induction with respect to the degree of $P \in S(m)$ in order to prove that $P \in I_{\text{mod}}(m)$ if $\lambda(P) \in \mathbb{D}_{H,\chi,\text{mod}}(G)$ (apply Lemma 2.7). The second implication in the proposition follows from Theorem 2.6(d). \square

Suppose for the moment that χ extends to a character on G and remember the mapping $f \mapsto \tilde{f}$ defined by (2.8). For $u \in \mathbb{D}_{H,\chi,\text{mod}}(G)$ define an operator D_u acting on $C^\infty(G/H)$ by

$$(2.17) \quad (D_u f)^\sim := u \tilde{f}, \quad f \in C^\infty(G/H).$$

Then $\text{supp}(D_u f) \subset \text{supp}(f)$, hence, by Peetre's theorem (cf. for instance NARASIMHAN [7, §3.3]), D_u is a differential operator on G/H . One easily shows that $D_u \in \mathbb{D}(G/H)$, the space of G -invariant differential operators on G/H .

THEOREM 2.9. Suppose that χ extends to a character on G . Then the mapping

$$u \Big|_{C_{H,\chi}^\infty(G)} \xrightarrow{C} D_u : \mathbb{D}_{H,\chi,\text{mod}}(G) \Big|_{C_{H,\chi}^\infty(G)} \xrightarrow{C} \mathbb{D}(G/H)$$

is an algebraic isomorphism onto.

PROOF. Clearly, C is an isomorphism into. In order to prove the surjectivity let $D \in \mathbb{D}(G/H)$. Then there is a polynomial $P \in S(m)$ such that

$$(Df)(g.0) = P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}\right) f(g \exp(t_1 X_1 + \dots + t_r X_r).0) \Big|_{t_1 = \dots = t_r = 0}$$

for all $f \in C^\infty(G/H)$ and for $g = e$. By the G -invariance of D this formula holds for all $g \in G$. By (2.8) and (2.2) this becomes

$$\chi(Df)^\sim = \lambda(P)(\chi \tilde{f}), \quad \text{i.e.}$$

$$(\mathrm{Df})^\sim = (\chi^{-1}\lambda(P) \circ \chi)(\tilde{f}).$$

Clearly, $\chi^{-1}\lambda(P) \circ \chi \in \mathbb{D}(G)$ and, by Lemma 2.5, we have

$$\chi^{-1}\lambda(P) \circ \chi \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G).$$

Thus, by (2.17):

$$D = D_{\chi^{-1}\lambda(P) \circ \chi}. \quad \square$$

Suppose now that the coset space G/H is *reductive*, i.e., m can be chosen such that $\mathrm{Ad}(h)m \subset m$ for all $h \in H$. From now on assume that m is chosen in this way. Let

$$(2.18) \quad \mathbb{D}_H(G) := \{D \in \mathbb{D}(G) \mid \mathrm{Ad}(h)D = D \text{ for all } h \in H\},$$

$$(2.19) \quad I(m) := \{P \in S(m) \mid \mathrm{Ad}(h)P = P \text{ for all } h \in H\}.$$

Then

$$\lambda(S(m)) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \lambda(I(m)) \subset \mathbb{D}_H(G).$$

Hence (2.13) becomes

$$(2.20) \quad \mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \mathbb{D}(G)h^\chi \oplus \lambda(I(m)).$$

We obtain from Theorems 2.6 and 2.9:

THEOREM 2.10. *Let G/H be reductive. Then:*

- (a) $\mathbb{D}_H(G)$ is a subalgebra of $\mathbb{D}(G)$.
- (b) $\mathbb{D}(G)h^\chi \cap \mathbb{D}_H(G)$ is a two-sided ideal in $\mathbb{D}_H(G)$.
- (c) There is a direct sum decomposition

$$(2.21) \quad \mathbb{D}_H(G) = \mathbb{D}(G)h^\chi \cap \mathbb{D}_H(G) \oplus \lambda(I(m)).$$

(d) Define the mappings A, B and C (C only if χ extends to a character on G) by

$$u \xrightarrow{A} u(\text{mod } \mathbb{D}(G)h^\chi \cap \mathbb{D}_H(G)) \xrightarrow{B} u \Big|_{C_{H,\chi}^\infty(G)} \xrightarrow{C} D_u:$$

$$\lambda(I(m)) \xrightarrow{A} \mathbb{D}_H(G)/(\mathbb{D}(G)h^\chi \cap \mathbb{D}_H(G)) \xrightarrow{B} \mathbb{D}_H(G) \Big|_{C_{H,\chi}^\infty(G)} \xrightarrow{C} \mathbb{D}(G/H).$$

Then A is a linear bijection and B and C are algebra isomorphisms onto.

The case $\chi \equiv 1$ of Theorem 2.10 can be found in HELGASON [4, Cor. X.2.6 and Theor. X.2.7]. See DUFLO [1] for the general case.

3. APPLICATION TO $\mathbb{D}(G/N)$ AND $\mathbb{D}(G/MN)$

Let G be a connected noncompact real semisimple Lie group. We remember some of the structure theory of G (cf. [3, Ch. VI]):

g_0 : Lie algebra of G.

g : complexification of g_0 .

θ : Cartan involution of g_0 , extended to automorphism of g .

$g_0 = k_0 + p_0$: Corresponding Cartan decomposition of g_0 .

h_{p_0} : maximal abelian subspace of p_0 , A the corresponding analytic subgroup.

h_0 : maximal abelian subalgebra of g_0 extending h_{p_0} .

$hk_0 := h_0 \cap k_0$, h_k its complexification.

h : complexification of h_0 ; this is a Cartan subalgebra of g .

Δ : set of roots of g with respect to h ; the roots are real on $ih_{k_0} + h_{p_0}$.

Introduce compatible orderings on $h_{p_0}^*$ and $(ih_{k_0} + h_{p_0})^*$.

Δ^+ : set of positive roots.

P_+ : set of positive roots not vanishing on h_{p_0} .

P_- : set of positive roots vanishing on h_{p_0} .

g^α : root space in g of $\alpha \in \Delta$.

n : $\sum_{\alpha \in P_+} g^\alpha$.

$n_0 := n \cap g_0$

N : analytic subgroup of G corresponding to n_0 .

M : centralizer of h_{p_0} in G , M_0 its identity component.
 m_0 : Lie algebra of M .
 m : complexification of m_0 ; then

$$(3.1) \quad m = h_k + \sum_{\alpha \in P_-} (g^\alpha + g^{-\alpha}).$$

PROPOSITION 3.1. *The coset spaces G/MN and G/N are not reductive.*

PROOF. Suppose that G/MN is reductive. Then there is an $\text{ad}_g(m+n)$ -invariant subspace \mathfrak{h} of \mathfrak{g} complementary to $m+n$. Let $\alpha \in P_+$ and let X be a nonzero element of g^α . For $H \in \mathfrak{h}$ write $H = W_H + Y_H + Z_H$ with $W_H \in \mathfrak{h}$, $Y_H \in m$, $Z_H \in n$. Then, for each $H \in \mathfrak{h}$:

$$\alpha(H)X = [W_H + Y_H + Z_H, X]$$

so

$$\alpha(H)X - [Y_H, X] - [Z_H, X] = [W_H, X] \in \mathfrak{h} \cap (m+n),$$

so

$$[Y_H, X] + [Z_H, X] = \alpha(H)X.$$

It follows from (3.1) that

$$[Y_H, X] + [Z_H, X] \in \sum_{\substack{\beta \in \Delta \\ \beta \neq \alpha}} g^\beta.$$

Hence $\alpha(H)X = 0$ for all $H \in \mathfrak{h}$, so $\alpha = 0$. This is a contradiction.

In the case G/N the proof is almost the same: take \mathfrak{h} $\text{ad}_g(n)$ -invariant and complementary to n and $Y_H = 0$. \square

HELGASON [5, p. 676] states without proof that G/MN is not in general reductive.

Let \mathfrak{l}_0 be the orthoplement of m_0 in \mathfrak{k}_0 with respect to the Killing form on \mathfrak{g}_0 . In order to apply Proposition 2.8 and Theorem 2.9 to $\text{ID}(G/MN)$ and

$\mathbb{D}(G/N)$ we take $\ell_0 + h_{p_0}$ respectively $k_0 + h_{p_0}$ as complementary subspaces of $m_0 + n_0$ respectively n_0 in g_0 . Now we have:

$$(3.2) \quad I_{\text{mod}}(\ell_0 + h_{p_0}) = S(h_{p_0}),$$

$$(3.3) \quad I_{\text{mod}}(k_0 + h_{p_0}) = S(m_0 + h_{p_0}).$$

(3.2) is proved in HELGASON [5, Lemma 4.2] and by only slight modifications in this proof, (3.3) is obtained. It follows from Lemma 2.5 that

$$\lambda(S(h_{p_0})) \subset \mathbb{D}_{MN,1,\text{mod}}(G)$$

and

$$\lambda(S(m_0 + h_{p_0})) \subset \mathbb{D}_{N,1,\text{mod}}(G),$$

since M centralizes h_{p_0} and $m_0 + h_{p_0}$ normalizes n_0 . Consider $\mathbb{D}(A)$ and $\mathbb{D}(M_0A)$ as subalgebras of $\mathbb{D}(G)$. Then $\mathbb{D}(A) \subset \mathbb{D}_{MN,1,\text{mod}}(G)$ and $\mathbb{D}(M_0A) \subset \mathbb{D}_{N,1,\text{mod}}(G)$. It follows by application of Proposition 2.8 and Theorem 2.9 that:

THEOREM 3.2. *The mapping $u \rightarrow D_u$ (cf. (2.17)) is an algebra isomorphism from $\mathbb{D}(A)$ onto $\mathbb{D}(G/MN)$ and from $\mathbb{D}(M_0A)$ onto $\mathbb{D}(G/N)$. In particular, $\mathbb{D}(G/MN)$ is a commutative algebra.*

The statements about $\mathbb{D}(G/MN)$ are in HELGASON [5, Theorem 4.1]. FARAUT [2, p. 393] observes that Helgason's result can be extended to the context of pseudo-Riemannian symmetric spaces.

A special case of Theorem 6.2 can be formulated in the situation that G is a connected complex semisimple Lie group. Let g be its (complex) Lie algebra and put:

- u : compact real form of g .
- a : maximal abelian subalgebra of u .
- $h := a + ia$; this is a Cartan subalgebra of g .
- Δ : set of roots of g with respect to h .

Δ^+ : set of positive roots with respect to some ordering.

g^α : root space of $\alpha \in \Delta$.

$n := \sum_{\alpha \in \Delta^+} g^\alpha$, N the corresponding analytic subgroup.

$g^{\mathbb{R}} := g$ considered as real Lie algebra.

$h^{\mathbb{R}} := h$ considered as real subalgebra

Then $g^{\mathbb{R}} = u + ia + n$ is an Iwasawa decomposition for $g^{\mathbb{R}}$ (cf.

[4, Theorem VI.6.3]) and a is the centralizer of ia in u . Hence we obtain from Theorem 3.2:

THEOREM 3.3. *The mapping $P \mapsto D_\lambda(P)$ is an algebra isomorphism from $S(h^{\mathbb{R}})$ onto $ID(G/N)$. In particular, $ID(G/N)$ is commutative.*

This theorem was proved by HELGASON [6, Lemma 3.3] without use of Theorem 3.2.

4. APPLICATION TO ISOTROPIC SPACES

We preserve the notation and conventions of Section 2. First we prove an extension of [4, Cor. X.2.8] to the case that G/H is not necessarily reductive. In the following, A and B are as in Theorem 2.6(d).

LEMMA 4.1. *If the algebra $I_{\text{mod}}(m)$ is generated by P_1, \dots, P_ℓ and if there are $Q_1, \dots, Q_\ell \in S_m$ such that $\text{degree}(P_i - Q_i) < \text{degree } P_i$ and $\lambda(Q_i) \in ID_{H, \chi, \text{mod}}(G)$ then the algebra*

$$ID_{H, \chi, \text{mod}} \Big|_{C_{H, \chi}^\infty(G)}$$

is generated by $BA\lambda(Q_1), \dots, BA\lambda(Q_\ell)$.

PROOF. We prove by complete induction with respect to m that, for each $P \in S_m(m)$ with $\lambda(P) \in ID_{H, \chi, \text{mod}}(G)$, $BA\lambda(P)$ depends polynomially on $BA\lambda(Q_1), \dots, BA\lambda(Q_\ell)$. In view of Theorem 2.6 this will prove the lemma. Suppose the above property holds up to $m-1$. Let $P \in S_m(m)$ such that $\lambda(P) \in ID_{H, \chi, \text{mod}}(G)$. By using Lemma 2.7 we find that $P = \Pi(P_1, \dots, P_\ell) \pmod{S_{m-1}(m)}$ for some polynomial Π in ℓ indeterminates. Hence, $P = \Pi(Q_1, \dots, Q_\ell) \pmod{S_{m-1}(m)}$,

$$\begin{aligned}
\lambda(P) &= \lambda(\Pi(Q_1, \dots, Q_\ell)) \pmod{\lambda(S_{m-1}(m))} \\
&= \Pi(\lambda(Q_1), \dots, \lambda(Q_\ell)) \pmod{\lambda(S_{m-1}(g))}, \\
\lambda(P) - \Pi(\lambda(Q_1), \dots, \lambda(Q_\ell)) &\in \lambda(S_{m-1}(g)) \cap \mathbb{D}_{H, \chi, \text{mod}}(G).
\end{aligned}$$

By Theorem 2.6 and formula (2.10) we have

$$\text{BA}\lambda(P) - \Pi(\text{BA}\lambda(Q_1), \dots, \text{BA}\lambda(Q_\ell)) = \text{BA}\lambda(P')$$

for some $P' \in S_{m-1}(m)$ such that $\lambda(P') \in \mathbb{D}_{H, \chi, \text{mod}}(G)$. Now apply the induction hypothesis. \square

Let τ denote the action of G on G/H . Its differential $d\tau$ yields an action of H on the tangent space $(G/H)_0$ to G/H at 0 .

THEOREM 4.2. *Suppose there is a nondegenerate $d\tau(H)$ -invariant bilinear form $\langle \cdot, \cdot \rangle$ on $(G/H)_0$ of signature (r_1, r_2) ($r_1 + r_2 = r$, $r_1 \geq r_2$) such that, for each $\lambda > 0$, $d\tau(H)$ acts transitively on $\{X \in (G/H)_0 \mid \langle X, X \rangle = \lambda\}$ (or on the connected components of these hyperbolas if $r_1 = r_2 = 1$). Let Δ be the Laplace-Beltrami operator on G/H corresponding to the G -invariant pseudo-Riemannian structure on G/H associated with $\langle \cdot, \cdot \rangle$. Then the algebra $\mathbb{D}(G/H)$ is generated by Δ , and hence commutative.*

PROOF. Choose a complementary subspace m to h in g . The mapping $d\pi$ identifies the H -spaces m (under $\sigma \circ \text{Ad}_G(H)$) and $(G/H)_0$ (under $d\tau(H)$) with each other. Transplant the form $\langle \cdot, \cdot \rangle$ to m and choose an orthonormal basis X_1, \dots, X_r of m : $\langle X_i, X_j \rangle = \varepsilon_i \delta_{ij}$, $\varepsilon_i = 1$ or -1 for $i \leq r_1$ or $> r_1$, respectively. Then the algebra $\mathbb{I}_{\text{mod}}(m)$ is generated by $\sum_{i=1}^r \varepsilon_i X_i^2$. It follows from the proof of Theorem 2.9 that $\Delta = D_{\lambda(P)}$ with $P \in S(m)$ of degree 2 such that $\lambda(P) \in \mathbb{D}_{H, 1, \text{mod}}(G)$. Thus, by Lemma 2.7, we get

$$P = c \sum_{i=1}^r \varepsilon_i X_i^2 \pmod{S_1(m)}$$

with $c \neq 0$. Now apply Lemma 4.1 and Theorem 2.9. \square

Theorem 4.2 extends [4, Prop. X.2.10], where the case is considered that G/H is a Riemannian symmetric space of rank 1. A pseudo-Riemannian manifold M is called *isotropic* if for each $x \in M$ and for tangent vectors $X, Y \neq 0$ at x with $\langle X, X \rangle = \langle Y, Y \rangle$ there is an isometry of M fixing x which sends X to Y . Connected isotropic spaces can be written as homogeneous spaces G/H satisfying the conditions of Theorem 4.2 with G being the full isometry group (cf. WOLF [8, Lemma 11.6.6]). It follows from Wolf's classification [8, Theorem 12.4.5] that such spaces are symmetric and reductive. However, our proof of Theorem 4.2 does not use this fact.

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